Electric Potential Energy of a Point Charge in an Electric Field

Consider moving a point charge from $\vec{r}_1$ to $\vec{r}_2$ along a contour $\mathcal{C}$. The work done *on* the charge is given by doing the line integral of the *negative* of the electric force along the path because that is the mechanical force that has to be exerted to move the charge against the electric force $\vec{F}_e$:

$$W_{12} = -\int_{\mathcal{C},\vec{r}_1}^{\vec{r}_2} d\vec{\ell} \cdot \vec{F}_e(\vec{r}) \tag{2.63}$$

The force is related to the electric field, and so we have

$$W_{12} = -q \int_{\mathcal{C},\vec{r}_1}^{\vec{r}_2} d\vec{\ell} \cdot \vec{E}(\vec{r}) = q\,[V(\vec{r}_2) - V(\vec{r}_1)] \tag{2.64}$$

That is, the work done on the charge by the mechanical force in going from $\vec{r}_1$ to $\vec{r}_2$ is given by the charge times the change in electric potential between the two positions. Note the sign: if the potential is higher at the end point, then the work done was positive.

Of course, this lets us to define the *electric potential energy* by

$$\boxed{U(\vec{r}_2) - U(\vec{r}_1) = q\,[V(\vec{r}_2) - V(\vec{r}_1)]} \tag{2.65}$$

That is, the electric potential energy of the charge and the electric potential of the field are simply related. Since it was defined in terms of work done against a force, electric potential energy obviously has units of Joules (J). That is explicit in the above form, which is C (N m/C) = (N m) = J.

Note that the electric field can also do work on the charge. In this case, the sign in the above line integral for the work is flipped and work is done as the charge loses potential energy. In this case, the work done by the electric field on a charge is what gives it the kinetic energy it has at the end.

Therefore, we replace $\mathcal{V}$ with all of space and let $\mathcal{S}$ go to infinity:

$$U = \frac{\epsilon_o}{2} \int_{r \to \infty} da\, \widehat{n} \cdot \left[V(\vec{r})\, \vec{E}(\vec{r}) \right] + \frac{\epsilon_o}{2} \int_{all\ space} |\vec{\nabla} V(\vec{r})|^2 \qquad (2.70)$$

Because the charge distribution is restricted to the finite volume $\mathcal{V}$ and thus looks like a point charge as $r \to \infty$, the field and potential fall off like $1/r^2$ and $1/r$. The surface area of $\mathcal{S}$ only grows as r^2, so the integral goes like $1/r$ and thus vanishes as $r \to \infty$. (If the charge distribution is not restricted to a finite volume, the surface term may not vanish and its potential energy could indeed be infinite!)

It may seem strange that we can make this choice of $\mathcal{S}$, as changing $\mathcal{V}$ and $\mathcal{S}$ affects both integrals in the last expression. The explanation is that the choice of $\mathcal{S}$ changes the two integrals but leaves their sum constant, and taking $\mathcal{S}$ to infinity simply zeros out the first integral, leaving only the contribution of the second integral.

We thus find

$$\boxed{U = \frac{\epsilon_o}{2} \int |\vec{E}(\vec{r})|^2} \qquad (2.71)$$

where the integral is over all of space. Correspondingly, the quantity $u = \frac{\epsilon_o}{2} |\vec{E}|^2$ is an energy density. We interpret this form as indicating that the potential energy created by assembling the charge distribution is stored in the field: less charge implies a smaller field and therefore less potential energy.

Superposition and Electric Potential Energy

Because the electric potential energy is a quadratic function of the electric field,

electric potential energy does not obey superposition

The energy of a sum of fields is more than just the sum of the energies of the individual fields because there is a cross term due to potential energy of the presence of the charges sourcing the second field in the first field (or vice versa, or half of both).

Electric Potential Energy of a Charge Distribution

How much work must be done to assemble a distribution of charge? This is most easily understood by first considering the assembly of a set of point charges one-by-one by bringing them in from infinity. When the ith charge is brought in, work must be done against the electric field of the first $i-1$ charges. Put another way, the ith charge starts with zero potential energy and ends with potential energy

$$U_i = \sum_{j=1}^{i-1} q_i \, \frac{1}{4\pi\epsilon_o} \frac{q_j}{|\vec{r}_i - \vec{r}_j|} \tag{2.66}$$

Thus, the total potential energy is

$$\boxed{U = \frac{1}{4\pi\epsilon_o} \sum_{i=1}^{N} \sum_{j=1}^{i-1} \frac{q_i q_j}{|\vec{r}_i - \vec{r}_j|} = \frac{1}{8\pi\epsilon_o} \sum_{i,j=1, i\neq j}^{N} \frac{q_i q_j}{|\vec{r}_i - \vec{r}_j|}} \tag{2.67}$$

where the factor of $1/2$ was introduced to allow i and j to both run from 1 to N. Generalizing this to a continuous charge distribution, we have

$$\boxed{U = \frac{1}{8\pi\epsilon_o} \int_{\mathcal{V}} d\tau \int_{\mathcal{V}} d\tau' \, \frac{\rho(\vec{r})\,\rho(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|}} \tag{2.68}$$

Electric Potential Energy in Terms of the Electric Field

We can use the relations between potential, field, and charge density (Equations 2.6, 2.53, and 2.61) and the divergence theorem (Equation 2.20) to obtain an alternate expression for the electric potential energy in terms of the electric field as follows:

$$\begin{aligned}
U &= \frac{1}{8\pi\epsilon_o} \int_{\mathcal{V}} d\tau \int_{\mathcal{V}} d\tau' \, \frac{\rho(\vec{r})\,\rho(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|} = \frac{1}{2} \int_{\mathcal{V}} d\tau \rho(\vec{r})\, V(\vec{r}) = -\frac{\epsilon_o}{2} \int_{\mathcal{V}} d\tau \left[\nabla^2 V(\vec{r})\right] V(\vec{r}) \\
&\overset{ibp}{=} -\frac{\epsilon_o}{2} \int_{\mathcal{V}} d\tau \, \vec{\nabla} \cdot \left[V(\vec{r})\, \vec{\nabla} V(\vec{r})\right] + \frac{\epsilon_o}{2} \int_{\mathcal{V}} |\vec{\nabla} V(\vec{r})|^2 \quad \text{with } \overset{ibp}{=} \equiv \text{integration by parts} \\
&\overset{\substack{\text{divergence} \\ \text{theorem}}}{=} \frac{\epsilon_o}{2} \int_{\mathcal{S}(\mathcal{V})} da \, \widehat{n} \cdot \left[V(\vec{r})\, \vec{E}(\vec{r})\right] + \frac{\epsilon_o}{2} \int_{\mathcal{V}} |\vec{\nabla} V(\vec{r})|^2
\end{aligned} \tag{2.69}$$

In the last line, the first term is an integral of the product of the potential and the field at the surface of the volume. In order to get the full energy of the charge distribution, $\mathcal{V}$ must include all the charge. If we assume the charge distribution is restricted to some finite volume, then $\mathcal{V}$ is naturally the volume containing the charge distribution. But we can add volume that does not contain charge because it contributes nothing to the initial expression for the electric potential energy.